Theory

Suppose we have an object that can exchange heat with its surroundings. If it is initially at a temperature $T$ higher than the environment temperature $T_e$, then it will expel heat and cool off, approaching equilibrium. This lab is about that approach to equilibrium, and it illustrates an empirical rule about the rate of cooling that was originally suggested by Newton.

Many quantities change more quickly when they are large than when they are small. Suppose $X(t)$ is a positive quantity that decreases toward zero with time. It may happen that

$$\frac{dX}{dt} = -bX.$$  \hspace{1cm} (1)

That is, the rate of decrease of $X$ is proportional to $X$ itself. (The constant coefficient $b > 0$ determines the exact relationship.) The solution to this differential equation is

$$X(t) = X_0 \, e^{-bt},$$ \hspace{1cm} (2)

where $X_0$ is the initial value of $X$. This general form is sometimes called "exponential decay". (You should be able to see why the exponential function is a solution to the Equation (1).) If we obtain $X$ data as a function of time, we will get a graph that looks something like the left-hand graph below.

However, it is hard to tell just by looking whether we have exponential decay or some other type of downward-swooping curve for $X(t)$. Hence, it makes sense to take the natural logarithm of $X$ and graph it as a function of time. Then our equation becomes

$$\ln X(t) = \ln X_0 - bt$$ \hspace{1cm} (3)

which is a linear relation! The resulting graph will have slope $-b$ and intercept $\ln X_0$, as shown in the right-hand graph above. A straight-line graph of $\ln X$ vs. $t$ is characteristic of exponential decay.
Exponential decay occurs in many different contexts, from the voltage across a discharging capacitor to the decay of a radioactive substance. It may also arise when a warm object approaches thermal equilibrium with its surroundings.

Let $E$ be the energy content of an object that is warmer than its surroundings. Isaac Newton suggested that the rate of heat loss from the object is proportional to the difference between the object's temperature and that of its environment. That is,

$$\frac{dE}{dt} = -A(T - T_0).$$  \hspace{1cm} (4)

Here $A$ is a constant that measures how readily heat is lost from the object. A small change $dE$ in the energy is related to a small change $dT$ in temperature by $dE = CdT$, where $C$ is the heat capacity of the object. Therefore

$$C \frac{dT}{dt} = -A(T - T_0).$$  \hspace{1cm} (5)

Let $\theta(t) = T(t) - T_0$, the difference between the object's temperature and the temperature of the environment. Then we can see that

$$\frac{d\theta}{dt} = -\left(\frac{A}{C}\right) \theta(t),$$  \hspace{1cm} (6)

whose solution is $\theta(t) = \theta_0 e^{-bt}$, where $b = \frac{A}{C}$. The temperature of the object should approach the environment via exponential decay. The "decay constant" $b$ will be smaller if the object is well insulated ($A$ small) or very large ($C$ large).

A characteristic of exponential decay is the half-life $t_{1/2}$, the time required for the decaying quantity to be reduced to one-half of its original value. This is related to the decay constant by $bt_{1/2} = \ln 2$.

**Experiment: Approach to equilibrium**

In this lab we will take advantage of the fact that the thermometer itself is an object, and its approach to room temperature is an example of Newton's Law of Cooling. As you carry out these steps, record your procedure, data and analysis (including answers to all questions asked) in your laboratory notebook.

1. Heat your thermometer up by immersing its tip in a container of hot water (temperature above 50°C). Remove the thermometer and quickly dry it, then suspend it in the air.

2. Use a clock or a stopwatch to keep track of the time. Every 10 seconds, record the temperature reading on the thermometer. Continue doing this for 4 minutes.
3. Wait for several more minutes, until the thermometer reading does not change during a whole minute. Record this reading; this is $T_e$, the temperature of the environment.

4. Enter your time and temperature data into Igor. Give your columns descriptive labels like "tim" and "temp". Copy the "temp" column and paste it into two more columns. These will be appear with automatic names like "temp1" and "temp2". (Note: All we are doing at this stage is setting aside new columns with the right number of data points. We'll modify the contents of these columns below.)

5. Rename the third column (your first new one) "theta". In the command line at the bottom of the screen, enter "theta = tim - $T_e$ ", where you replace $T_e$ with the measured value you got in step 1. The "theta" column should now contain the differences between your temperature readings and room temperature.

6. Create a graph of $\theta$ vs. $t$. Be sure to adjust the scale of the $\theta$ axis to include $\theta = 0$. What is the general shape of this graph? Make an "eyeball estimate" of $t_{1/2}$ from your data.

7. Rename your fourth column "logtheta". In the command line at the bottom of the screen, enter "logtheta = ln(theta)". The "logtheta" column should now contain the natural logarithm of the temperature difference. (Note: You should print out a data table that shows the theta and logtheta calculated values, and put it in your lab notebook.)

8. Create a graph of ln($\theta$) vs. $t$. Is this graph linear? Fit a straight line to this data. (Note: Do not worry about the units for ln($\theta$). It is unitless. The logarithm function effectively "annihilates" all units.)

9. Is your temperature decrease an example of exponential decay? From your analysis, determine the decay constant $b$ and the half-life $t_{1/2}$ for your cooling thermometer. Does the half-life nearly agree with the "eyeball estimate" from step 6? Report your values in standard (Kenyon) form.

10. Answer these questions as part of your lab notebook discussion: How would the experimental results from step 9 be different if the thermometer had been immersed in a container of room-temperature water instead of cooling in air? How would they be different if room temperature had been only 0º C? (That would be a chilly lab!)