2.19 If $N$ is the number of dipoles, the multiplicity of the state with $n$ spins down is (from Eqn. 2.7):

$$\Omega(N,n) = \binom{N}{n} = \frac{N!}{n!(N-n)!}.$$ 

Let’s take the natural logarithm and use that form of Stirling’s approximation.

$$\ln \Omega(N,n) = \ln N! - \ln n! - \ln (N-n)!$$
$$= (N \ln N - N) - (n \ln n - n) - ((N-n) \ln (N-n) - (N-n))$$
$$= N \ln N - n \ln n - (N-n) \ln (N-n)$$
$$= \ln \frac{N^n}{n^n (N-n)^{N-n}}$$

If we exponentiate both sides, we get an approximate expression for the multiplicity of the two state paramagnet.

$$\Omega(N,n) \approx \frac{N^n}{n^n (N-n)^{N-n}}.$$ 

Now we are asked to further simplify this expression in the limit that $n / N \to 0$ (the low temperature limit). We go back a couple of steps and use the trick we learned in problem 2.13 b):

$$\ln \Omega \approx N \ln N - n \ln n - (N-n) \ln (N-n)$$
$$= N \ln N - n \ln n - (N-n) \ln \left[ N \left( 1 - \frac{n}{N} \right) \right]$$
$$= N \ln N - n \ln n - (N-n) \ln N \frac{n}{N}$$
$$= N \ln N - n \ln n - N \ln n + N \ln N + n \left( 1 + \frac{n}{N} \right)$$
$$= -n \ln n + n \ln N + n$$

Thus,

$$\Omega \approx \left( \frac{Ne}{n} \right)^n.$$ 

In the low temperature limit of the Einstein solid, there are very few energy units per oscillator. Thus the odds of there being more than one energy unit in a given oscillator is small. Hence the multiplicity of the Einstein oscillator in this limit must approach a problem in which there is a binary choice about whether there is or is not an energy unit in a particular oscillator. But that is analogous to the choice of whether a spin is up or down. Hence the two problems are isomorphic in the low temperature limit.