PHYS 218 – Spring, '07 – Assignment #1 – Radioactive Decay Chains

In this assignment, we will simulate what happens when each of the following isotopes is allowed to decay for 4.5 Gy (10^9 years): ${}^{238}_{92}U$, ${}^{235}_{92}U$, and ${}^{232}_{90}Th$. Each of you will be assigned to one of the isotopes for simulation. This project will ask you to produce code in C or C++ to remind you of and increase your programming skill.

This simulation, as is typical in a "real world" investigation, will require a number of phases with each stage well documented:

1) *Basic physics*: To spread the work out a bit, I will assign each of you to research only one of the radioactive decay chain of ${}^{238}_{92}U$, ${}^{235}_{92}U$, or ${}^{232}_{90}Th$. A "decay chain" is the sequence of isotopes produced by the original isotope until it reaches a stable isotope which is the last isotope in the chain. Much information is available on the web and to do a simulation we will have to have details of which isotopes decay into which other isotopes and the radioactive half-lives of each decay. I recommend starting with http://ie.lbl.gov/education/isotopes.htm. Ignore branches in the chain with probability (*branching ratio*) less than 5%.

Product 1.1: A table in MS word format clearly showing your assigned decay chain including all half-lives. Make sure your results check with that available on the Web and with another class member who was assigned the same original isotope.

2) *Preliminary coding exercise #1*: We need to develop algorithms to simulate the process of radioactive decay. The fundamental law of radioactive decay is,

$$\frac{dN(t)}{dt} = -\lambda N(t)$$

where *N(t)* is the number of nuclei of a given type as a function of time and the decay constant is $\lambda = \ln 2 / t_{1/2}$, with $t_{1/2}$ the half life of the isotope. It is often more convenient to specify the activity of a radioactive isotope, but that is an easy conversion since $A(t) = \lambda N(t)$. The SI unit of activity is the *becquerel (Bq)* which is one decay each second. A more common unit in use in the US is the *curie (Ci)* which is $3.7 \text{x} 10^{10}$ Bq.

Product 1.2: Write two programs in *C* or *C++* that tracks the activity as a function of time in increments of 10 My of 1 Ci of the first isotope in your assigned chain for a total time of 1 Gy by solving the differential equation above. The first program should use the Euler method and the second program should use the fourth-order Runge-Kutta method. You should compare each program's output with the exact solution of the differential equation,

$$A(t) = A(t = 0)e^{-\lambda t}$$

and adjust the time steps in your program so that the error in the activity is never larger than 0.01%. Present evidence of the superiority in speed of the fourth-order Runge-Kutta method and document your results in a MS Word file which includes listings of your programs, a data table, and graphs of your data.

3) *Preliminary coding exercise #2.* If we are going to model a chain, we have to know how the numbers of isotopes relate to each other. Consider a three isotope chain,

$$\frac{dN_1}{dt} = -\lambda_1 N_1$$
$$\frac{dN_2}{dt} = -\lambda_2 N_2 + \lambda_1 N_1$$
$$\frac{dN_3}{dt} = +\lambda_2 N_2$$

with the last isotope assumed stable. Convince yourself that you understand these equations and can generalize them to a complete chain. Now here is the issue. If you look at the chains we want to simulate we can see there are huge disparities in the half-lives of the various isotopes. If we proceed to simulate the chain in a straightforward way, we would have to take time steps small compared to the shortest half-life. You can see that this would then take an enormous number of steps to simulate a few billion years of time. So, we want to come up with a shortcut.

Product 1.3: Write a program in *C* or *C++* that simulates the three isotope chain. Use fourth-order Runge Kutta. First use it to simulate the case when all the decay constants are equal to 1 and $N_1(0) = 1$, $N_2(0) = 0$, and $N_3(0) = 0$. Produce a graph of all the $N_i(t)$ from $t = 0$ to $t = 5$. Now explore the case where $\lambda_2 << \lambda_1$. Use your simulation to show how we can both eliminate this step in the chain for times much longer than $1/\lambda_2$ and still know the activity of isotope N_2. Document your results in an MS Word report.

4) *Final coding exercise:* Simulate the time evolution of the numbers and activity of each element in your radioactive decay chain in increments of 10 My for a total time of 4.5 Gy.

Product 1.4: Simulate the time evolution using a C or C++ program that uses fourth-order Runge Kutta. In an MS Word document, present your results, including a discussion of any approximations you make, a listing of your program, a data table, and plots of your data. Compare the results to the relative abundances of each isotope found in meteoric rocks.

Potential end-of-semester project that continues these ideas:
Simulate the time evolution of lead (Pb) isotopes in rocks that make up the Earth. The relative abundances of Pb isotopes changes as a function of time because Pb isotopes are the stable endpoints of the radioactive decay chains that start at higher values of atomic number (*Z*) and mass number *(A)*. The basic idea is that heavy isotopes are created in supernova explosions with high values of both *Z* and *A*, then the unstable isotopes decay radioactively, and at some time are incorporated into the rocks that make up planets, moons, and meteorites in the Solar System. By examining the relative abundances of Pb isotopes in today's rocks, you will be able to establish a date for the creation of the rocks.