Due: Friday, 10/24, 2:10pm, PHYS360 Assignment 8v2

Reading 1. Griffiths, Ch.3, pg. 114-124. Prepare for Reading Quiz: Friday 10/24, questions on anything in Griffiths, pages 114-124, but especially:

a) understand Dirac notation, 118-120
b) be able to represent operators by the matrix elements, pg 120
c) understand Example 3.8, pg 120 on the two-state system
   bra \( |\psi_1\rangle \) ket
   dual space
   projection operator
d) be able to define:
ed) understand the “tidiest” way to express completeness, pg 123

1. In some orthonormal basis an operator \( \hat{T} = |\phi_1\rangle\langle\phi_1| + 2|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1| \). Find the matrix, representing \( \hat{T} \).

2. An important operator used in quantum computation is the “Hadamard gate,” which is represented by the matrix:
   \[
   H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
   \]

   Show whether \( H \) is Hermitian and unitary. (Recall that for a unitary matrix \( U \), \( UU^\dagger = 1 \), or equivalently \( U^\dagger = U^{-1} \))

3. Suppose that in some orthonormal basis \( |u_1\rangle, |u_2\rangle, |u_3\rangle \) an operator \( \hat{A} \) acts as follows:
   \[
   \hat{A}|u_1\rangle = 2|u_1\rangle \\
   \hat{A}|u_2\rangle = 3|u_1\rangle - i|u_3\rangle \\
   \hat{A}|u_3\rangle = -|u_2\rangle
   \]

   Write the matrix representation of the operator.

4. Is the following set of vectors linearly independent?
   \[
   |a\rangle = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} , |b\rangle = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} , |c\rangle = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}
   \]

   Hint: Recall that for linearly independent vectors, the equation:
   \[
   a_1|a\rangle + a_2|b\rangle + a_3|c\rangle = 0
   \]

   can only be true if \( a_1 = a_2 = a_3 = 0 \).
An orthonormal basis of a Hamiltonian operator in four dimensions is defined as follows

\[ H|1\rangle = E|1\rangle, H|2\rangle = 2E|2\rangle, H|3\rangle = 3E|3\rangle, H|4\rangle = 4E|4\rangle \]

A system is in the state

\[ |\psi\rangle = 3|1\rangle + 2|2\rangle - |3\rangle + 7|4\rangle \]

a) If a measurement of the energy is made, what results can be found and with what probabilities? (Check normalization)
b) Find the average energy of the system.

On a side (refer to the book’s appendix for full treatment): A vector is specified with respect to a a certain basis \( |e_n\rangle \), with components \( a^e_i \). In another basis \( |f_n\rangle \), the same vector will have different components, say \( a^f_i \). These components transform as

\[ a^f_i = \sum_{j=1}^{n} S_{ij} a^e_j \]

where \( S_{ij} \) are the components of the matrix \( S \) that transforms \( |f_n\rangle \) into \( |e_n\rangle \).

Now suppose there is a matrix \( T^e \) that transforms any vector specified in the \( |e_n\rangle \) basis to a different vector (that has also components specified in the same basis.)

Question: What are the components of the matrix in the \( |f_n\rangle \) basis \( T^f \) that does the same transformation?

Answer: \( T^f = S T^e S^{-1} \)

Example:

The kets \( |\psi_1\rangle \) and \( |\psi_2\rangle \) form an orthonormal basis. We define a new basis \( |\phi_1\rangle \) and \( |\phi_2\rangle \) by

\[ |\phi_1\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle + |\psi_2\rangle) \]
\[ |\phi_2\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle - |\psi_2\rangle) \]

a) Calculate the elements of the matrix \( S_{ij} \) that transforms the basis.
b) Find \( S^{-1} \)
c) An operator \( \hat{P} \) is represented in the \( |\psi_i\rangle \)-basis by the matrix \( P = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \).

What are the components of \( P \) in the \( |\phi_i\rangle \) basis? (While there are several ways of doing this problems, you must use \( T^f = S T^e S^{-1} \))

Question: Is there a way to transform any matrix \( T^e \) (the superscript \( e \) indicates the components are with respect to \( |e_n\rangle \) basis) to a basis in which the matrix \( T^f \) in the new basis is diagonal (with only diagonal elements nonzero)? In other words, can we find \( S \) (the matrix that transforms the basis) such that \( T^f = S T^e S^{-1} \), where \( T^f \) is diagonal? The answer is yes; follow this procedure (you are finding a similarity transformation for a matrix \( T^e \)).
1. Find the eigenvalues and eigenvectors of the matrix $T^e$
2. Normalize the eigenvectors of $T^e$
3. Form a new matrix $S$ by forming the columns of $S$ with the eigenvectors of $T^e$

In quantum mechanics, we are concerned primarily with Hermitian and Unitary matrices. In that case, the diagonal transformation of a hermitian matrix $H$ takes the form:

$$T^f = U^\dagger HU$$

This holds because $U^\dagger = U^{-1}$ for a unitary matrix. Such a transformation is called a unitary transformation.

Example:

7. Consider the matrix $R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, the $2 \times 2$ rotation matrix.

Find a unitary transformation that diagonalizes $R$ (Hint: a) Find the eigenvalues and eigenvectors of $R$ b) normalize the eigenvectors. c) Construct the matrix $U$ by having the two eigenvectors of $R$ as columns of $U$. d) Write down $U^\dagger$. e) Check that $UU^\dagger = I$. f) Finally, calculate $U^\dagger RU$. Is the final matrix diagonal?