**Due**: Friday, 10/24, 2:10pm, PHYS360 Assignment 8v2

**Reading** 1. Griffiths, Ch.3, pg. 114-124. Prepare for Reading Quiz: Friday 10/24, questions on anything in Griffiths, pages 114- 124, but especially:

a) understand Dirac notation, 118-120

b) be able to represent operators by the matrix elements, pg 120

c) understand Example 3.8, pg 120 on the two-state system

d) be able to define:
- bra
- ket
- dual space
- projection operator

e) understand the "tidiest" way to express completeness, pg 123

**1** In some orthonormal basis an operator $\hat{T} = |\phi_1\rangle\langle\phi_1| + 2|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|$. Find the matrix, representing $\hat{T}$.

**2** An important operator used in quantum computation is the "Hadamard gate," which is represented by the matrix:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Show whether $H$ is Hermitian and unitary. (Recall that for a unitary matrix $U$, $UU^\dagger = 1$, or equivalently $U^\dagger = U^{-1}$)

**3** Suppose that in some orthonormal basis $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$ an operator $\hat{A}$ acts as follows:

$$\hat{A}|u_1\rangle = 2|u_1\rangle$$
$$\hat{A}|u_2\rangle = 3|u_1\rangle - i|u_3\rangle$$
$$\hat{A}|u_3\rangle = -|u_2\rangle$$

Write the matrix representation of the operator.

**4** Is the following set of vectors linearly independent?

$$|a\rangle = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, |b\rangle = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, |c\rangle = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}$$

Hint: Recall that for linearly independent vectors, the equation:

$$a_1|a\rangle + a_2|b\rangle + a_3|c\rangle = 0$$

can only be true if $a_1 = a_2 = a_3 = 0$.

**5** An orthonormal basis of a Hamiltonian operator in four dimensions is defined as follows

$$H|1\rangle = E|1\rangle, H|2\rangle = 2E|2\rangle, H|3\rangle = 3E|3\rangle, H|4\rangle = 4E|4\rangle$$

A system is in the state

$$|\psi\rangle = 3|1\rangle + |2\rangle - |3\rangle + 7|4\rangle$$

a) If a measurement of the energy is made, what results can be found and with what probabilities? (Check normalization) b) Find the average energy of the system.

On a side (refer to the book's appendix for full treatment): A vector is specified with respect to a a certain basis $|e_n\rangle$, with components $a_i^e$. In another basis $|f_n\rangle$, the same vector will have different components, say $a_i^f$. These components transform as $a_i^f = \sum_{i=1}^n S_{ij} a_i^e$ where $S_{ij}$ are the components of the matrix $\mathbf{S}$ that transforms $|f_n\rangle$ into $|e_n\rangle$.

Now suppose there is a matrix $\mathbf{T}^e$ that transforms any vector specified in the $|e_n\rangle$ basis to a different vector (that has also components specified in the same basis.)

Question: What are the components of the matrix in the $|f_n\rangle$ basis $\mathbf{T}^f$ that does the same transformation?

Answer: $\mathbf{T}^f = \mathbf{S}\mathbf{T}^e\mathbf{S}^{-1}$

Example:

**6** The kets $|\psi_1\rangle$ and $|\psi_2\rangle$ form an orthonormal basis. We define a new basis $|\phi_1\rangle$ and $|\phi_2\rangle$ by

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle)$$

a) Calculate the elements of the matrix $S_{ij}$ that transforms the basis.

b) Find $\mathbf{S}^{-1}$

c) An operator $\hat{P}$ is represented in the $|\psi_i\rangle$-basis by the matrix $\mathbf{P} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$. What are the components of $\mathbf{P}$ in the $|\phi_i\rangle$ basis? (While there are several ways of doing this problems, you must use $\mathbf{T}^f = \mathbf{S}\mathbf{T}^e\mathbf{S}^{-1}$)

Question: Is there a way to transform any matrix $\mathbf{T}^e$ (the superscript $e$ indicates the components are with respect to $|e_n\rangle$ basis) to a basis in which the matrix $\mathbf{T}^f$ in the new basis is diagonal (with only diagonal elements nonzero)? In other words, can we find $\mathbf{S}$ (the matrix that transforms the basis) such that $\mathbf{T}^f = \mathbf{S}\mathbf{T}^e\mathbf{S}^{-1}$, where $\mathbf{T}^f$ is diagonal? The answer is yes; follow this procedure (you are finding a similarity transformation for a matrix $\mathbf{T}^e$):

1. Find the eigenvalues and eigenvectors of the matrix $\mathbf{T}^e$
2. Normalize the eigenvectors of $\mathbf{T}^e$
3. Form a new matrix $\mathbf{S}$ by forming the columns of $\mathbf{S}$ with the eigenvectors of $\mathbf{T}^e$

In quantum mechanics, we are concerned primarily with Hermitian and Unitary matrices. In that case, the diagonal transformation of a hermitian matrix $\mathbf{H}$ takes the form:

$$\mathbf{T}^f = \mathbf{U}^\dagger \mathbf{H} \mathbf{U}$$

This holds because $\mathbf{U}^\dagger = \mathbf{U}^{-1}$ for a unitary matrix. Such a transformation is called a *unitary transformation.*

Example:

**7** . Consider the matrix $\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, the $2 \times 2$ rotation matrix. Find a unitary transformation that diagonalizes $\mathbf{R}$ (Hint: a) Find the eigenvalues and eigenvectors of $\mathbf{R}$ b) normalize the eigenvectors. c) Construct the matrix $\mathbf{U}$ by having the two eigenvectors of $\mathbf{R}$ as columns of $\mathbf{U}$. d) Write down $\mathbf{U}^\dagger$. e) Check that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$. f) Finally, calculate $\mathbf{U}^\dagger\mathbf{R}\mathbf{U}$. Is the final matrix diagonal?